COMM 616: Modern Optimization with Applications in ML and OR	**2024-25 Fall**

Assignment I

Instructor: Jiajin Li — Due: October 14th, 2024

Problem 1 (5 pt). *Let* $\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x}^T\mathbf{A}\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{x} + c \leq 0\}$, *where* $\mathbf{A} \in \mathcal{S}^d, \boldsymbol{b} \in \mathbb{R}^d$, *and* $c \in \mathbb{R}$ *are given.*

(i) *Show that* $\mathcal{S}$ *is convex if* $\mathcal{A}$ *is positive semi-definite. Is the converse true? Explain.*

(ii) *Let* $\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{g}^T\boldsymbol{x} + h = 0\}$, *where* $\boldsymbol{g} \in \mathbb{R}^d \backslash \{\mathbf{0}\}$ *and* $h \in \mathbb{R}$. *Show that* $\mathcal{S} \cap \mathcal{H}$ *is convex if* $\mathcal{A} + \lambda \boldsymbol{g}\boldsymbol{g}^T$ *is postive semidefinite for some* $\lambda \in \mathbb{R}$.

Problem 2 (5 pt). *Let* $\mathbf{C} \in \mathcal{S}^d_+$ *be given. Show that the function* $f : \mathcal{S}^d_{++} \to \mathbb{R}_+$ *given by* $f(\mathbf{X}) = tr(\mathbf{C}\mathbf{X}^{-1})$ *is convex.*

Problem 3 (10 pt). *Given a set* $\mathcal{C} \subset \mathbb{R}^d$, *define the indicator function* $\mathbb{1}_{\mathcal{C}} : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *of* $\mathcal{C}$ *by*

$$\mathbb{1}_{\mathcal{C}}(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \boldsymbol{x} \in \mathcal{C} \\ +\infty & \text{otherwise} \end{cases}$$

Compute $\mathbb{1}^*_{\mathcal{C}}$, *the conjugate of* $\mathbb{1}_{\mathcal{C}}$, *for the following sets. Show your calculations.*

(i) $\mathcal{C} = \left\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{a}^T\boldsymbol{x} \leq b\right\}$, *where* $\boldsymbol{a} \in \mathbb{R}^d \backslash \{\mathbf{0}\}$ *and* $b \in \mathbb{R}$ *are given.*

(ii) $\mathcal{C} = \left\{\boldsymbol{x} \in \mathbb{R}^d_+ : \|\boldsymbol{x}\|_2 \leq 1\right\}$.

Problem 4 (10 pt). *Let* $g : \mathbb{R}^n \to \mathbb{R}, h : \mathbb{R}^m \to \mathbb{R}$ *be given convex functions,* $\mathbf{A} \in \mathbb{R}^{m \times n}$ *be a given matrix and* $\boldsymbol{b} \in \mathbb{R}^m, \boldsymbol{c} \in \mathbb{R}^n$ *be two given vectors. Consider the following problems:*

$$\begin{aligned} v_p^* &= \inf_{\boldsymbol{x} \in \mathbb{R}^n} \left\{\boldsymbol{c}^T\boldsymbol{x} + h(\boldsymbol{b} - \mathbf{A}\boldsymbol{x}) + g(\boldsymbol{x})\right\} \\ v_d^* &= \sup_{\boldsymbol{y} \in \mathbb{R}^m} \left\{\boldsymbol{b}^T\boldsymbol{y} - h^*(\boldsymbol{y}) - g^*(\mathbf{A}^T\boldsymbol{y} - \boldsymbol{c})\right\}. \end{aligned}$$

Here, $f^*(\boldsymbol{z}) = \sup_{\boldsymbol{x} \in \mathbb{R}^n} \left\{\boldsymbol{z}^T\boldsymbol{x} - f(\boldsymbol{x})\right\}$ *is the conjugate of* f. *Use the three methods discussed in class—Lagrangian duality, Fenchel-Rockafellar duality, and perturbation methods—to demonstrate that* $v_p^* \geq v_d^*$.

Problem 5 (10 pt). *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a continuously differentiable convex function and* $\mathcal{C} \subset \mathbb{R}^d$ *be a non-empty closed convex set. Show that* $x^\star$ *is an optimal solution to the convex optimization problem*

$$\min_{\boldsymbol{x} \in \mathcal{C}} f(\boldsymbol{x})$$

if and only if

$$\boldsymbol{x}^\star = \text{proj}_{\mathcal{C}}\left(\boldsymbol{x}^\star - \nabla f\left(\boldsymbol{x}^\star\right)\right)$$

where $\text{proj}_{\mathcal{C}}(\cdot)$ *is the projection operator onto* $\mathcal{C}$.